Infinite products. Weierstrass' canonical products.

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<u>Def</u> Let (p_n) be a sequence of complex numbers. We say that $\prod_{n=1}^{\infty} p_n$ converges if $\exists \lim_{N\to\infty} \prod_{n=1}^{N} p_n \neq 0.$ ($\neq 0$ not to create extra zeroes) $p_n \neq 0$.

<u>Lemma.</u> If $\prod_{n=1}^{\infty} p_n$ converges, then $p_n \to 1$.

<u>Proof.</u> Let $P_N := \prod_{n=1}^{N} p_n$. Let $P_N \to P$.

$$p_n = \frac{P_n}{P_{n-1}} \to \frac{P}{P} = 1. \blacksquare$$

<u>Remark.</u> The opposite is not true.

<u>Example</u>. $p_n = \frac{n+1}{n} \to 1$, but $P_N = N+1 \to \infty$ ∎

<u>Theorem.</u> $\prod_{n=1}^{\infty} p_n$ converges if and only if

$S := \sum_{n=1}^{\infty} \operatorname{Log} p_n$ converges.

Moreover, if convergence holds, $\prod_{n=1}^{\infty} p_n = e^S$.

<u>Proof.</u> $S_N := \sum_{n=1}^{N} \operatorname{Log} p_n$

$P_N = e^{S_N} = \prod_{n=1}^{N} p_n$.

(!!) We are <u>not</u> saying that $\operatorname{Log} P_N \to \operatorname{Log} P$!

If $S_N \to S$ then, by continuity of exponent, $e^{S_N} \to e^S$, i.e. $P_N \to e^S \in \mathbb{C}\setminus\{0\}$

<u>Conversely</u>: let $P_N \to P \in \mathbb{C}\setminus\{0\}$.

By multiplying p_1 by $e^{-i\operatorname{Arg} P}$, we can assume that $P > 0$.

Then $\operatorname{Log} P_N \to \operatorname{Log} P$ (Log is continuous at P).

$p_n \to 1$, so $\operatorname{Log} p_n \to 0$.

Observe: $\operatorname{Log} P_N = \left(\sum_{n=1}^{N} \operatorname{Log} p_n\right) + 2\pi i k_N$ for some $k_N \in \mathbb{Z}$.

$$2\pi i (k_N - k_{N-1}) = \underbrace{\operatorname{Log} P_N - \operatorname{Log} P_{N-1}}_{\to 0} - \underbrace{\operatorname{Log} p_N}_{\to 0},$$

so $\exists N_0 : N \geq N_0 \quad k_N = k_{N_0}$.

So $\sum_{n=1}^{N} \operatorname{Log} p_j = \operatorname{Log} P_N + 2\pi i k_{N_0} \to \operatorname{Log} P + 2\pi i k_{N_0}$ – converges! ∎

<u>Def.</u> $p_n \in \mathbb{C}\setminus\{0\}$, $\prod_{n=1}^{\infty} p_n$ converges absolutely if $\sum |\operatorname{Log} p_n| < \infty$.

<u>Lemma.</u> $p_n \in \mathbb{C}\setminus\{0\}$ $\prod_{n=1}^{\infty} p_n$ converges absolutely if and only if $\sum_{n=1}^{\infty} |p_n - 1|$ converges.

<u>Proof.</u> Observe: $\lim_{z\to 1} \frac{\operatorname{Log} z}{z-1} = \lim_{z\to 1} \frac{\operatorname{Log} z - \operatorname{Log} 1}{z-1} = \frac{1}{z}\Big|_{z=1} = 1.$

Since if either $\prod p_n$ or $\sum |p_n - 1|$ converge, $p_n \to 1$, we have in any of the cases $\frac{|\operatorname{Log} p_n|}{|p_n - 1|} \xrightarrow[n\to\infty]{} 1$.

So $\exists N : n \geq N \quad 2|p_n - 1| \geq |\operatorname{Log} p_n| \geq \frac{1}{2}|p_n - 1|$.

so $\frac{1}{2}\sum_{n=N}^{\infty} |p_n - 1| \leq \sum_{n=N}^{\infty} |\operatorname{Log} p_n| \leq 2\sum_{n=N}^{\infty} |p_n - 1|$ – converge simultaneously ∎

<u>Remark.</u> Not true for non-absolute convergence.

$\prod_{n=1}^{\infty} p_n$ – converge $\Longleftrightarrow$ $\sum (p_n - 1)$ converge.

Take $p_n = e^{(-1)^n/\sqrt{n}}$. Then $\sum \log p_n = \sum \frac{(-1)^n}{\sqrt{n}}$ – converges (alternating series) but $\sum(p_n-1)=\sum\left(e^{(-1)^n i/\sqrt{n}}-1\right)=\sum\left(1-\frac{(-1)^n}{\sqrt{n}}+\frac{1}{2}\frac{1}{n}+\dots-1\right)=$

$\underbrace{\sum\frac{(-1)^{n+1}}{\sqrt{n}}}_{\text{converges}}+\underbrace{\sum\frac{1}{2n}}_{\text{diverges}}+\underbrace{\text{higher order}}_{\text{converges}}$.

If $p_n=1+i\frac{(-1)^n}{\sqrt{n}}$, then $\sum p_n-1=\sum\frac{(-1)^n}{\sqrt{n}}$ converges,
but $\sum \operatorname{Log} p_n$ diverges.

Bonus (+2pts) Prove it rigorously.

Products of functions.

Def. Let $f_n:\Omega\to\mathbb{C}$ - a sequence of functions.
$\prod_{n=1}^{\infty} f_n(z)$ converges locally uniformly on Ω
if $\forall$ compact $K\subset\Omega$ $\exists N(K)$: for $n\geq N(K)$, $f_n(z)\neq 0$ $\forall z\in K$
and $\prod_{n=N(K)}^{\infty} f_n$ converges uniformly and absolutely on K.

$f_n(z)\in\mathcal{A}(\Omega)$

Remark. If $f:=\prod_{n=1}^{\infty} f_n(z)$ converges locally uniformly in Ω then
$\forall$ compact $K\subset\Omega$ $f=\underbrace{\prod_{n=1}^{N(K)-1} f_n(z)}_{\in\mathcal{A}(\Omega)}\cdot\underbrace{\prod_{n=N(K)}^{\infty} f_n(z)}_{\text{converges uniformly}}\in\mathcal{A}(K)$. So $f\in\mathcal{A}(\Omega)$.

Also $f(z)=0 \iff \exists n:\ f_n(z)=0$
$\operatorname{ord}(f,z)=\underbrace{\sum_k \operatorname{ord}(f_n,z)}_{\text{finite sum}}$ Indeed $\underset{\{z\}\text{ - compact}}{\Rightarrow}\prod_{n=N(\{z\})}^{\infty} f_n(z)$ converges, so $\neq 0$.

So $f(z)=0 \iff f(z)=\prod_{n=1}^{N(\{z\})-1} f_n(z)\ g(z)$, where $g(z)\neq 0$.

First, we study Weierstrass factors:

$E_0(z):=1-z$, $E_n(z):=(1-z)\exp\left(z+\frac{z^2}{2}+\dots+\frac{z^n}{n}\right)$ $n\geq 1$ $E_n(z)=0\iff z=1$ $\operatorname{ord}(E_n(z),1)=1$.

Motivation: $\log E_n(z)=\log(1-z)+z+\frac{z^2}{2}+\dots+\frac{z^n}{n}=-z-\frac{z^2}{2}\ \dots-\frac{z^n}{n}-\frac{z^{n+1}}{n+1}-\dots+z+\dots+\frac{z^n}{n}=$
$-\underbrace{\sum_{k=n+1}^{\infty}\frac{z^k}{k}}\simeq O(|z|^{n+1})$.

Observe. $E_n'(z)=(1-z)(1+z+\dots+z^{n-1})\exp\left(z+\frac{z^2}{2}+\dots+\frac{z^n}{n}\right)-\exp\left(z+\frac{z^2}{2}+\dots+\frac{z^n}{n}\right)=$
$=(1-z^n-1)\exp\left(z+\frac{z^2}{2}+\dots+\frac{z^n}{n}\right)=-z^n\exp\left(z+\dots+\frac{z^n}{n}\right)$.

Claim. If $|z|\leq 1$ then
$|E_n(z)-1|\leq |z|^{n+1}$

Proof.
Observe: $|E_n'(tz)|=|tz|^n\exp\left(tz+\dots+\frac{t^nz^n}{n}\right)=|z|^n\left(t^n\exp\left(tz+\dots+\frac{t^nz^n}{n}\right)\right)\leq$
$\leq -|z|^n E_n'(t)$ if $|z|\leq 1$ (since $|t^kz^k|=t^k|z^k|\leq t^k$)

$E_n(z)-E_n(0)=z\int_0^1 E_n'(tz)\,dt$ so

$|E_n(z)-1|\leq |z|\int_0^1 |E_n'(tz)|\,dt\leq -|z|^{n+1}\int_0^1 E_n'(t)\,dt=-|z|^{n+1}(E_n(1)-E_n(0))=|z|^{n+1}$

$E_n(z) - E_n(0) = z\int_0^1 E_n'(tz)\,dt$ so

$$|E_n(z) - 1| \le |z| \int_0^1 |E_n'(tz)|\,dt \le -|z|^{n+1} \int_0^1 E_n'(t)\,dt = -|z|^{n+1}(E_n(1) - E_n(0)) = |z|^{n+1}$$ ∎

Lemma. Let $z_n \to \infty$, and (k_n) be a sequence of natural numbers such that $\sum_{n=1}^{\infty} \left(\frac{r}{|z_n|}\right)^{k_n+1} < \infty$ $\forall r > 0$.

Then $\prod_{n=1}^{\infty} E_{k_n}(z/z_n)$ converges locally uniformly. In particular, $\prod_{n=1}^{\infty} E_{k_n}(z/z_n) \in A(\mathbb{C})$.

Weierstrass product.

Proof

If K is compact, $\exists r: K \subset B(0,r)$.

So, since $z_n \to \infty$, $\exists N: n \ge N$ $|z_n| \ge r$. So $n \ge N \Rightarrow z_n \notin K$.

Note now that for $z \in K$:

$$|E_{k_n}(z/z_n) - 1| \le \left|\frac{z}{z_n}\right|^{k_n+1} \le \left(\frac{r}{|z_n|}\right)^{k_n+1}.$$

So $$\sum_{n=N}^{\infty} |E_{k_n}(z/z_n) - 1| \le \sum \left(\frac{r}{|z_n|}\right)^{k_n+1} < \infty$$ ∎

Remark. $k_n = n$ works for any $z_n \to \infty$:

$$\sum \frac{r^{n+1}}{|z_n|^{n+1}} < \infty \ \forall r, \ \left(\lim \sqrt[n+1]{\frac{1}{|z_n|^{n+1}}} = \lim_{n\to\infty} \left(\frac{1}{|z_n|}\right)^{\frac{n}{n+1}} = 0. \right).$$

But for some sequences, can do better!

Theorem (Weierstrass Product Theorem).
Let $z_n \to \infty$. Then there exists an entire function f, such that $f(z) = 0 \Leftrightarrow z = z_n$ for some n.
The order of zero at each z is equal to the number of times z appear in (z_n) ($\operatorname{ord}(f,z) = \#\{n : z_n = z\}$).

Proof. Let $k = \#\{n : z_n = 0\}$.

Define

$$f(z) = z^k \prod_{\substack{n=1 \\ z_n \ne 0}}^{\infty} E_n\left(\frac{z}{z_n}\right) = z^k \prod_{\substack{n=1 \\ z_n \ne 0}}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n} + \frac{1}{2}\frac{z^2}{z_n^2} + \dots + \frac{1}{n}\frac{z^n}{z_n^n}\right)$$

By Lemma, $f(z) \in A(\mathbb{C})$. $f(z) = 0 \Leftrightarrow \exists n: z_n = z$.

$$\operatorname{Ord}(f,z) = \sum \operatorname{ord}\left(E_n\left(\frac{z}{z_n}\right), z\right) + \operatorname{Ord}(z^k, z) = \begin{cases} k, & z = 0 \\ \#\{n : z_n = z\} & z \ne 0 \end{cases}$$ ∎

Theorem (Weierstrass factorization theorem).
Let f be entire. Then

$$f = e^{g(z)} z^m \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n} + \frac{1}{2}\left(\frac{z}{z_n}\right)^2 + \dots + \frac{1}{k_n}\left(\frac{z}{z_n}\right)^{k_n}\right)$$

where $m = \operatorname{ord}(f,0)$, z_n - zeroes of f, listed as many times as their multiplicities, $g \in A(\mathbb{C})$.

Proof. Let (z_n) be zeroes of f, listed as many times as their multiplicities. $z_n \to \infty$, $m = \operatorname{ord}(f,0)$.
For any (k_n): $\sum \left(\frac{r}{|z_n|}\right)^{k_n+1} < \infty$ $(\forall r)$ we have

$$f_0(z) = z^m \prod_{n=1}^{\infty} E_{k_n}\left(\frac{z}{z_n}\right) \in A(\mathbb{C})$$

and $h(z) := \frac{f(z)}{f_0(z)} \in A(\mathbb{C})$ ($\in A(\mathbb{C} \setminus \{z_n, n \in \mathbb{N}\})$ - obvious removable singularities at z_n)

$f_0(z) = z^m \prod_{n=1} E_{k_n}\left(\frac{z}{z_n}\right) \in \mathcal{A}(\mathbb{C})$

and $h(z) := \frac{f(z)}{f_0(z)} \in \mathcal{A}(\mathbb{C})$ ($\in \mathcal{A}(\mathbb{C} \setminus \{z_n, n \in \mathbb{N}\})$ - obvious removable singularities at z_n.)

$h(z) \neq 0$, $\mathbb{C}$ - simply connected $\Rightarrow \exists g : h = e^g$ ∎

<u>Theorem</u>. (Interpolation Theorem)

Let $z_n \to \infty$, $z_n \neq z_m$ $(n \neq m)$. Let $(w_n)_{n=1}^{\infty} \subset \mathbb{C}$.

Then $\exists f \in \mathcal{A}(\mathbb{C}) : f(z_n) = w_n \ \forall n$.

<u>Proof</u>. Let $g \in \mathcal{A}(\mathbb{C})$,

g has simple zeroes at z_n and in no other points.

Then for each n, $\exists K_n(z) \in \mathcal{A}(\mathbb{C})$: $g(z) = (z - z_n) K_n(z)$,

$b_n := K_n(z_n) \neq 0$.

Let $h(z) \in \mathcal{M}(\mathbb{C})$, h has simple poles at (z_n),

and the singular part of h at z_n is $\frac{w_n/b_n}{z - z_n}$ - exists by Mittag-Leffler.

$h(z) = \frac{w_n/b_n}{z - z_n} + h_n(z)$, $h_n \in \mathcal{A}(B(z_n, \delta))$

Then $f(z) = g(z) h(z) \in \mathcal{A}(\mathbb{C})$ (poles at z_n cancel with zeroes),

and $\lim_{z \to z_n} f(z) = \lim_{z \to z_n} (z - z_n) k_n(z) \left(\frac{w_n/b_n}{z - z_n} + h_n(z) \right) = \frac{w_n}{b_n} k_n(z_n) = w_n$ ∎

<u>Theorem</u> (Quotient representation of meromorphic functions).

Let $h \in \mathcal{M}(\mathbb{C})$. Then $\exists f, g \in \mathcal{A}(\mathbb{C})$ without common zeroes, such that $h(z) = \frac{f(z)}{g(z)}$.

<u>Proof</u>. Let z_n be poles of h, listed with multiplicities.

g - entire function with poles only in z_n.

Then $f(z) = h(z) g(z) \in \mathcal{A}(\mathbb{C})$

<u>Theorem</u> (Root criterion).

Let $f \in \mathcal{A}(\mathbb{C})$. TFAE:

$k \geq 1$.

1) $\exists$ holomorphic kth root of f: $f = g^k$, $g \in \mathcal{A}(\mathbb{C})$

2) $\forall z \in \mathbb{C}$: $f(z) = 0 \Rightarrow k \mid \operatorname{ord}(f, z)$.

<u>Proof</u>. 1) $\Rightarrow$ 2) $\operatorname{ord}(f, z) = k \operatorname{ord}(g, z)$

2) $\Rightarrow$ 1) Let $z_1, \ldots, z_n, \ldots$ be zeroes, with multiplicities $k_1, \ldots k_n, \ldots$. Then let $\tilde{g} \in \mathcal{A}(\mathbb{C})$ with the same zeroes and multiplicities: k_1/k, k_2/k, ...

Then $h := \frac{f(z)}{\tilde{g}^n(z)} \in \mathcal{A}(\mathbb{C})$, $h(z) \neq 0 \Rightarrow \exists \hat{g} : \hat{g}^n = h$

Then take $g := \tilde{g} \hat{g}$, $g^n = \tilde{g}^n \hat{g}^n = f$ ∎

<u>Theorem</u>. (Weierstrass in a domain).

Let Ω be a region, (z_n) a sequence in Ω without limit points.

Then $\exists f \in \mathcal{A}(\Omega)$: $f(z) = 0 \Leftrightarrow z = z_n$ for some n,

$\mathrm{ord}(f,z) = \#\{n : z_n = z\}$

No proof.

Canonical products.

Let $z_n \to \infty$. Assume: $z_n \neq 0$

Claim. $\prod_{n=1}^{\infty} E_k(z/z_n)$ converges locally uniformly (Same k for all n).
if and only if $\sum_{n=1}^{\infty} \frac{1}{|z_n|^{k+1}} < \infty$.

$\left(1-\frac{z}{z_n}\right)\exp\left(z+\frac{z^2}{2z_n^2}+\ldots+\frac{z^k}{k z_n^k}\right)$

$1+\frac{z}{z_n^2}+\ldots+\frac{z^{k-1}}{z_n^k}$

Proof. $(\Uparrow)$ by Lemma (since $\sum_{n=1}^{\infty} \frac{1}{|z_n|^{k+1}} < \infty \iff \sum_{n=1}^{\infty}\left(\frac{r}{|z_n|}\right)^{k+1} = r^{k+1}\sum_{n=1}^{\infty}\frac{1}{|z_n|^{k+1}} < \infty$ $\forall r$).

$(\Downarrow)$ Let $f(z) := \prod_{n=1}^{\infty} E_k(z/z_n)$.

$$\left(\prod_{n=1}^{N} E_k(z/z_n)\right)' = \sum_{n=1}^{N}\left(E_k'(z/z_n)\prod_{j\neq n} E_k(z/z_j)\right)$$

$$g(z) := \frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty}\left(\frac{1}{(z-z_n)} + \sum_{j=1}^{k}\frac{z^{j-1}}{z_n^j}\right)$$ — converges uniformly and absolutely at $B(0,\delta)$ for some small δ.

$$\frac{P_N'}{P_N} = \sum \frac{E_k'(z/z_n)}{E_k(z/z_n)}$$

So the same is true for its kth derivative (by Weierstrass)

$$g^{(k)}(z) = \sum_{n=1}^{\infty}\frac{(-1)^k k!}{(z-z_n)^{k+1}}.$$ So this series converges uniformly at 0: $k!\sum\frac{1}{|z_n|^{k+1}} < \infty$ ∎

$P_N' \to f'$

$P_N \to f$

$\frac{P_N'}{P_N} \to \frac{f'}{f}$.

Def. Let $K = \min\{k : \sum \frac{1}{|z_n|^{k+1}} < \infty\}$.

Then $\prod E_K(z/z_n)$ is called the canonical product for (z_n)
K - genus of the canonical product.

Examples:

1) $F_q(z) := \prod_{n=1}^{\infty}(1+q^n z)$, where $|q|<1$.

Euler partition function.

Canonical product for $z_n = -q^{-n}$. Genus is 0.

$F_q(qz)(1+qz) = F_q(z)$

$$F_q(z) = 1 + \sum_{n=1}^{\infty}\frac{q^{\frac{n(n+1)}{2}}}{(1-q)\ldots(1-q^n)} z^n$$

Proof.

$F_q(z) = \sum_{n=0}^{\infty} a_n z^n$ $\quad a_0 = F_q(0) = 1$.

$F(qz)(1+qz) = F_q(z) \Rightarrow \left(\sum a_n q^n z^n\right)(1+qz) = \sum a_n z^n \Rightarrow$

$a_n q^n + a_{n-1}q^n = a_n \Rightarrow a_n = \frac{q^n}{1-q^n}a_{n-1} \underset{\text{induction}}{\Longrightarrow}$

$$\Rightarrow a_n = \frac{\prod_{k=1}^{n} q^k}{\prod_{k=1}^{n}(1-q^k)} = \frac{q^{\frac{n(n+1)}{2}}}{\prod_{k=1}^{n}(1-q^k)}.$$

2) $z_n = \pi n$ $\quad n \in \mathbb{Z}$.

$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{1}{|z_n|} = \infty$ but $\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{1}{|z_n|^2} < \infty$, so genus is 1.

$$z\prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(1-\frac{z}{\pi n}\right)e^{\frac{z}{\pi n}}$$

Can be simplified, if we group n and $-n$.

$$f(z) = z\prod_{n=1}^{\infty}\left(1-\left(\frac{z}{\pi n}\right)^2\right).$$

Let us compute $f(z)$.

Observe: $\sin z$ has the same zeroes, so

$\sin z = g(z)\, f(z)$, $\quad g \in \mathcal{A}(\mathbb{C})$, $\forall z,\ g(z) \neq 0$.

Observe: $f(z) = z \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{\pi n}\right)^2\right) = \lim_{N\to\infty} \underbrace{z \prod_{n=1}^{N} \left(1 - \left(\frac{z}{\pi n}\right)^2\right)}_{h_N(z)}$ uniformly on compacts.

By Weierstrass: $f'(z) = \lim_{N\to\infty} h_N'(z)$.

So $$\frac{f'(z)}{f(z)} = \lim_{N\to\infty} \frac{h_N'(z)}{h_N(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - (\pi n)^2} \overset{\text{unpair them}}{=} \frac{1}{z} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{1}{z - \pi n} + \frac{1}{\pi n}\right).$$

on any compact K such that $\forall n \in \mathbb{Z},\ \pi n \notin K$.

$$\cot z = \frac{\sin' z}{\sin z} = \frac{g'(z)}{g(z)} + \frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} + \frac{1}{z} + \sum_{n\neq 0} \left(\frac{1}{z - \pi n} + \frac{1}{\pi n}\right)$$

$$\| \quad \frac{1}{z} + \sum_{n\neq 0} \left(\frac{1}{z - \pi n} + \frac{1}{\pi n}\right)$$

So $g'(z) = 0 \Rightarrow g \equiv \text{const}$.

So $\sin z = C z \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{\pi n}\right)^2\right)$

But $1 = \lim_{z\to 0} \frac{\sin z}{z} = C \Rightarrow$

$$\boxed{\sin z = z \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{\pi n}\right)^2\right)}$$

3) $z_n = -n$, $n \in \mathbb{N} \setminus \{0\}$.

Again, genus is 1.

The canonical product

$$G(z) := z \prod_{n=1}^{\infty} E_1(z/n) = z \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}.$$

$-G(-z)$ - canonical product for $z_n = n$, $n \in \mathbb{N} \setminus \{0\}$.

$$-G(z)G(-z) = z^2 \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right)\left(1 + \frac{z}{n}\right) e^{-z/n} e^{z/n} = z^2 \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = \frac{z \sin \pi z}{\pi}.$$

(from product for sin.)

Claim $G(1) = e^{-\gamma}$, where $\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{n} - \log n\right)$ - Euler constant.

$\gamma \approx$ 0.5772156649015328606065120900824024310421...

Proof Observe: $\prod_{n=1}^{N} \left(1 + \frac{1}{n}\right) = N + 1$

So $G(1) = \lim_{N\to\infty} \prod_{n=1}^{N} \left(1 + \frac{1}{n}\right) e^{-\frac{1}{n}} = \lim_{N\to\infty} \exp\left(\log(N+1) - \sum_{n=1}^{N} \frac{1}{n}\right)$.

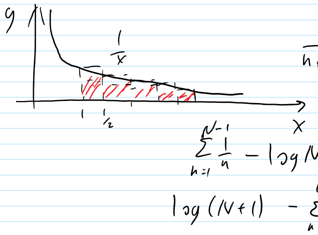
$G(1)$ exists, so $\lim_{N\to\infty} \log(N+1) - \sum_{n=1}^{N} \frac{1}{n}$ exists.

But $\lim_{N\to\infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{N} - \log N\right) = -\lim_{N\to\infty} \left(\log(N+1) - 1 - \dots - \frac{1}{N}\right) +$

$\lim_{N\to\infty} \log(N+1) - \log N$.

$\lim_{N\to\infty} \log\left(1 + \frac{1}{N}\right) = 0$

Another derivation of limit:



$$\frac{1}{n+1} < \int_n^{n+1} \frac{dx}{x} = \log \frac{n+1}{n} < \frac{1}{n}$$

So $\frac{1}{n} - \log \frac{n+1}{n} > 0 \Rightarrow$

$\sum_{n=1}^{N-1} \frac{1}{n} - \log N = \sum_{n=1}^{N-1} \left(\frac{1}{n} - \log \frac{n+1}{n}\right)$ is increasing in N.

$\log(N+1) - \sum_{n=2}^{N} \frac{1}{n} = \sum_{n=1}^{N} \left(\log \frac{n+1}{n} - \frac{1}{n+1}\right)$ is decreasing in N.

For $t \in \mathbb{R}, t > 0$, we define $t^z := e^{z \log t}$ ($\log$ - usual real logarithm).

Observe: $z \prod_{k=1}^{N} \left(1 + \frac{z}{n}\right) e^{-z/n} = \frac{z(z+1)\dots(z+N)}{N!\, N^z} \exp\left(z\left(\log N - \sum_{n=1}^{N} \frac{1}{n}\right)\right)$

so $\boxed{G(z) = e^{-\gamma z} \lim_{n \to \infty} \frac{z(z+1)\dots(z+n)}{n!\, n^z}}$